

# Pfaffian decomposition and a Pfaffian analogue of $q$ -Catalan Hankel determinants

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## Abstract

Motivated by the Hankel determinant evaluation of moment sequences, we study a kind of Pfaffian analogue evaluation. We prove an  $LU$ -decomposition analogue for skew-symmetric matrices, called Pfaffian decomposition. We then apply this formula to evaluate Pfaffians related to some moment sequences of classical orthogonal polynomials. In particular we obtain a product formula for a kind of  $q$ -Catalan Pfaffians. We also establish a connection between our Pfaffian formulas and certain weighted enumeration of shifted reverse plane partitions.

## 1 Introduction

The Hankel determinants of Catalan numbers have drawn the interests of many researchers with relations to the combinatorial arguments of lattice paths in recent years (see, e.g., [2, 5, 6, 11, 13, 21, 25]). It is well-known that

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if  $\{\mu_n\}_{n \geq 0}$  is the moment sequence of certain orthogonal polynomials, the Hankel determinant  $\det(\mu_{i+j-2})_{1 \leq i, j \leq n}$  have a nice formula because of the classical theory of orthogonal polynomials (see [13]). In this paper we would like to exploit a Pfaffian analogue of this kind of Hankel determinants.

We say a matrix  $A = (a_{ij}^i)_{i, j \geq 1}$  (resp.  $A = (a_{ij}^i)_{1 \leq i, j \leq n}$ ) is *skew-symmetric* if it satisfies  $a_i^j = -a_j^i$  for  $i, j \geq 1$  (resp.  $1 \leq i, j \leq n$ ). If we are given an  $n \times n$  skew-symmetric matrix  $A = (a_{ij}^i)_{1 \leq i, j \leq n}$  where  $n$  is an even integer, then the *Pfaffian* of  $A$  (see [25, 26]), denoted by  $\text{Pf } A$ , is defined to be

$$\text{Pf}(A) = \sum \epsilon(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n) a_{\sigma_2}^{\sigma_1} \dots a_{\sigma_n}^{\sigma_{n-1}}, \quad (1.1)$$

where the summation is over all partitions  $\sigma = \{\{\sigma_1, \sigma_2\}_<, \dots, \{\sigma_{n-1}, \sigma_n\}_<\}$  of  $[n] := \{1, 2, \dots, n\}$  into 2-elements blocks, and where  $\epsilon(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$  denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_{n-1} & \sigma_n \end{pmatrix}, \quad (1.2)$$

and we use the notation  $[n]$ . A partition  $\sigma$  of  $[n]$  into 2-elements blocks is called a *perfect matching* or a *1-factor*.

As most of the orthogonal polynomials have their  $q$ -analogues, in order to propose a Pfaffian analogue of the above Hankel determinants of moments, we have the ordinary version and  $q$ -version. More precisely, we propose  $\text{Pf}((j-i) \mu_{i+j+r-2})_{1 \leq i, j \leq 2n}$  as a Pfaffian analogue of the above Hankel determinants of the moments  $\mu_n$ , and for a  $q$ -analogue of  $\mu_n(q)$  of  $\mu_n$  we take

$$\text{Pf}((q^{i-1} - q^{j-1}) \mu_{i+j+r-2}(q))_{1 \leq i, j \leq 2n},$$

where  $r$  is a fixed integer. We mainly investigate the case where  $\mu_n$  is the moments of the little  $q$ -Jacobi polynomials in this paper. Throughout this paper we use the standard notation for  $q$ -series (see [4, 9]):

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}$$

for any integer  $n$ . Usually  $(a; q)_n$  is called the  *$q$ -shifted factorial*, and we frequently use the compact notation:

$$\begin{aligned} (a_1, a_2, \dots, a_r; q)_\infty &= (a_1; q)_\infty (a_2; q)_\infty \dots (a_r; q)_\infty, \\ (a_1, a_2, \dots, a_r; q)_n &= (a_1; q)_n (a_2; q)_n \dots (a_r; q)_n. \end{aligned}$$

The  ${}_{r+1}\phi_r$  *basic hypergeometric series* is defined by

$${}_{r+1}\phi_r \left[ \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n.$$

The *little  $q$ -Jacobi polynomials* [9, 19] are defined by

$$p_n(x; a, b; q) = \frac{(aq; q)_n}{(abq^{n+1}; q)_n} (-1)^n q^{\binom{n}{2}} {}_2\phi_1 \left[ \begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, xq \right], \quad (1.3)$$

which are orthogonal with respect to the *inner product* defined by

$$\langle f, g \rangle = \frac{(aq; q)_{\infty}}{(abq^2; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(bq; q)_k}{(q; q)_k} (aq)^k f(q^k) g(q^k). \quad (1.4)$$

The moments of the little  $q$ -Jacobi polynomials are defined by

$$\mu_n = \langle x^n, 1 \rangle = \frac{(aq; q)_n}{(abq^2; q)_n} \quad (n = 0, 1, 2, \dots).$$

The main results on  $\text{Pf} \left( (q^{i-1} - q^{j-1}) \mu_{i+j-2} \right)_{1 \leq i, j \leq 2n}$  are stated in Section 3.

To prove the Pfaffian identities we employ an  $LU$ -type decomposition of a skew-symmetric matrix, which we call a Pfaffian decomposition. In Section 2 we state this decomposition and give a proof by using a Pfaffian analogue of the Desnanot-Jacobi adjoint matrix theorem [7, Theorem 3.12]. In Section 4 we give a proof of our main results stated in Section 3. We prove the Pfaffian decomposition in Theorem 3.1 by reducing the single sum obtained as the matrix multiplication to the  $q$ -Dougall formula (4.10) for a terminating very-well-poised  ${}_6\phi_5$  series (see [4, 9]). As a byproduct of the proof we obtain the Pfaffian decomposition of another skew-symmetric matrix stated in Theorem 4.3.

As an application of our main results in Section 3 and Section 4, we obtain a formula for weighted enumeration of shifted reverse plane partitions in Section 5. We consider a special family of shifted reverse plane partitions and give weights that resembles to the weight in the inner product (1.4) to profiles of shifted reverse plane partitions in the family (see (5.4) and (5.20)). Then Corollary 3.2 (resp. Corollary 4.4) gives the weighted enumeration of the family of shifted reverse plane partitions whose number of rows are even (resp. odd).

In Section 6 we state several conjectures for this type of Pfaffians. One may ask what is the relation between our Pfaffians and the classical theory of orthogonal polynomials. At this point we have no answer to the question why the Pfaffians factors into nice linear factors from the view point of the classical theory.

Finally, in Appendix we state our second proof of the main results in Section 4 using Zeilberger's creative telescoping. We see that the certificates are simple, and we can prove the formulas by hand.

## 2 Pfaffian decomposition

First we recall the reader a well-known decomposition of a matrix. Let  $A = (a_j^i)_{i,j \geq 1}$  be a matrix (of finite or infinite row/column length). If  $I = \{i_1, \dots, i_r\}$  (resp.  $J = \{j_1, \dots, j_r\}$ ) are a set of row (resp. column) indices, then we write  $A_J^I = A_{j_1, \dots, j_r}^{i_1, \dots, i_r}$  for the  $r \times r$  submatrix obtained from  $A$  by choosing the rows indexed by  $I$  and columns indexed by  $J$ . Let  $a_J^I = a_{j_1, \dots, j_r}^{i_1, \dots, i_r}$  denote  $\det A_J^I$  if  $|I| = |J| > 0$ , and 1 if  $I = J = \emptyset$ . The following identity is known as the Desnanot-Jacobi adjoint matrix theorem [7, Theorem 3.12]

$$\begin{aligned} \det A_{[n-2]}^{[n-2]} \det A_{[n]}^{[n]} \\ = \det A_{[n-2], n-1}^{[n-2], n-1} \det A_{[n-2], n}^{[n-2], n} - \det A_{[n-2], n}^{[n-2], n-1} \det A_{[n-2], n-1}^{[n-2], n}. \end{aligned} \quad (2.1)$$

The following proposition is usually called the  $LU$ -decomposition of a matrix. Usually  $LU$ -decomposition means  $L$  is lower unitriangular and  $U$  is upper triangular. But here we adopt the style of  $LDU$ -decomposition where both of  $L$  and  $U$  are (lower or upper) unitriangular and  $D$  is diagonal because of Theorem 2.2. If  $A$  is an  $n \times n$  matrix of rank  $n$ , then, by elementary linear algebra, we can deduce that there is an  $n \times n$  permutation matrix  $P$  such that  $\det (PA)_{[i]}^{[i]} \neq 0$  for any  $i = 1, \dots, n$ . Although this permutation matrix  $P$  is not unique, the triple  $(L, D, U)$  in the following theorem is unique for chosen  $A$  and  $P$ . But this is the most general case, and in many applications we can choose  $P$  to be the identity matrix  $I_n$ . We give a proof here to make this paper more comprehensive and as a warm-up for the succeeding proof of Theorem 2.2.

**Proposition 2.1.** Let  $n$  be a positive integer, and  $A = (a_j^i)_{1 \leq i, j \leq n}$  be an  $n \times n$  matrix of rank  $n$ . If we choose an  $n \times n$  permutation matrix  $P$  such

that  $\det (PA)_{[i]}^{[i]} \neq 0$  for  $1 \leq i \leq n$ , then  $PA$  is uniquely written as

$$PA = L D U, \quad (2.2)$$

where  $D = (d_i \delta_j^i)_{1 \leq i, j \leq n}$  is a diagonal matrix,  $L = (l_j^i)_{1 \leq i, j \leq n}$  is a lower unitriangular matrix and  $U = (u_j^i)_{1 \leq i, j \leq n}$  is an upper unitriangular matrix. In fact

$$d_i = \frac{\det (PA)_{[i]}^{[i]}}{\det (PA)_{[i-1]}^{[i-1]}}, \quad l_j^i = \frac{\det (PA)_{[j]}^{[j-1], i}}{\det (PA)_{[j]}^{[j]}}, \quad u_j^i = \frac{\det (PA)_{[i-1], j}^{[i]}}{\det (PA)_{[i]}^{[i]}}.$$

Here the Kronecker delta  $\delta_j^i$  takes the value 1 if  $i = j$ , and 0 otherwise.

**Proof.** We may assume  $P = I_n$  without loss of generality, and we have to show that  $a_j^i$  is uniquely written as  $a_j^i = \sum_{k=1}^{\min(i, j)} l_k^i d_k u_j^k$  with  $l_i^i = u_i^i = 1$  for  $1 \leq i \leq n$ . This is trivial if  $n = 1$ . Assume  $n \geq 2$  and this is true for all  $1 \leq i, j \leq n-1$ . If  $1 \leq i < n$ , then  $u_n^i$  must satisfy

$$a_n^i = \sum_{k=1}^{i-1} \frac{a_{[k]}^{[k-1], i} a_{[k-1], n}^{[k]}}{a_{[k-1]}^{[k-1]} a_{[k]}^{[k]}} + d_i u_n^i.$$

But this can be obtained from

$$a_i^i = \sum_{k=1}^{i-1} \frac{a_{[k]}^{[k-1], i} a_{[k-1], i}^{[k]}}{a_{[k-1]}^{[k-1]} a_{[k]}^{[k]}} + \frac{a_{[i]}^{[i]}}{a_{[i-1]}^{[i-1]}}$$

by replacing  $i$ th column by  $n$ th column of  $A$ , and we obtain  $d_i u_n^i = a_{[i-1], n}^{[i]} / a_{[i-1]}^{[i-1]}$ .

Hence we derive  $u_n^i = a_{[i-1], n}^{[i]} / a_{[i]}^{[i]}$ , and vice versa. Similarly, when  $1 \leq j \leq n-1$ , we can show that  $l_j^n$  is determined uniquely and given by the above formula. Thus it is enough to prove the formula for  $i = j = n$ , which implies

$$a_n^n = \sum_{k=1}^{n-1} \frac{a_{[k]}^{[k-1], n} a_{[k-1], n}^{[k]}}{a_{[k-1]}^{[k-1]} a_{[k]}^{[k]}} + d_n.$$

By induction hypothesis

$$a_{n-1}^{n-1} = \sum_{k=1}^{n-2} \frac{a_{[k]}^{[k-1], n-1} a_{[k-1], n-1}^{[k]}}{a_{[k-1]}^{[k-1]} a_{[k]}^{[k]}} + \frac{a_{[n-1]}^{[n-1]}}{a_{[n-2]}^{[n-2]}},$$

which implies

$$a_n^n - \sum_{k=1}^{n-2} \frac{a_{[k]}^{[k-1],n} a_{[k-1],n}^{[k]}}{a_{[k-1]}^{[k-1]} a_{[k]}^{[k]}} = \frac{a_{[n-2],n}^{[n-2],n}}{a_{[n-2]}^{[n-2]}}.$$

Hence

$$d_n = a_n^n - \sum_{k=1}^{n-1} \frac{a_{[k]}^{[k-1],n} a_{[k-1],n}^{[k]}}{a_{[k-1]}^{[k-1]} a_{[k]}^{[k]}} = \frac{a_{[n-2],n}^{[n-2],n}}{a_{[n-2]}^{[n-2]}} - \frac{a_{[n-1]}^{[n-2],n} a_{[n-2],n}^{[n-1]}}{a_{[n-2]}^{[n-2]} a_{[n-1]}^{[n-1]}},$$

which equals  $a_{[n]}^{[n]}/a_{[n-1]}^{[n-1]}$  by (2.1). Conversely, if we take  $d_n = a_{[n]}^{[n]}/a_{[n-1]}^{[n-1]}$ , then it clearly satisfies the above equation and gives the  $LU$ -decomposition of  $A$ .  $\square$

The fact that each entry of  $L$ ,  $D$  and  $U$  is expressed with certain type of minors of  $A$  appears in [23] related to the Painlevé equations. Although we can use this decomposition even in the case where  $A$  is a skew-symmetric matrix, it seems more consistent to consider the decomposition in the following theorem in which each entry is expressed with subpfaffians. This type of decomposition also seems important with relation to the integrable systems (see [1]). Let us start with some definitions.

We define  $2 \times 2$  skew-symmetric matrix  $J_2$  by

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and let  $J_{2n} = J_2 \oplus \cdots \oplus J_2$  denote the  $2n \times 2n$  matrix whose main diagonal  $2 \times 2$  blocks are all  $J_2$  and the other blocks are  $2 \times 2$  zero matrices  $O_2$ . Note that  ${}^t J_{2n} J_{2n} = I_{2n}$ .

For a skew-symmetric matrix  $A$ , we usually take  $I = J$  so, hereafter, we write  $A_I = A_{i_1, \dots, i_r}$  for  $A_I^I$ . Further let  $a_I = a_{i_1, \dots, i_r}$  denote  $\text{Pf } A_I$  if  $I \neq \emptyset$ , 1 if  $I = \emptyset$  when there is no fear of confusion. Then the Pfaffian analogue of the Desnanot-Jacobi adjoint-matrix theorem (2.1) reads as follows (see [15, 17]):

$$\begin{aligned} a_{[n-4]} a_{[n]} &= a_{[n-4], n-3, n-2} a_{[n-4], n-1, n} \\ &\quad - a_{[n-4], n-3, n-1} a_{[n-4], n-2, n} + a_{[n-4], n-3, n} a_{[n-4], n-2, n-1}. \end{aligned} \tag{2.3}$$

The following theorem gives so-called *Pfaffian decomposition* of a skew-symmetric matrix  $A$ .

**Theorem 2.2.** Let  $n$  be a positive integer, and  $A = (a_j^i)_{1 \leq i, j \leq 2n}$  be a skew-symmetric matrix of size  $2n$ . If  $a_{[2i]} \neq 0$  for  $1 \leq i \leq n$ , then  $A$  is uniquely written as

$$A = {}^t V T V. \quad (2.4)$$

Here  $T$  and  $V$  are composed of  $2 \times 2$  blocks

$$T = \begin{pmatrix} T_1 & O_2 & \dots & O_2 \\ O_2 & T_2 & \dots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \dots & T_n \end{pmatrix}, \quad V = \begin{pmatrix} J_2 & V_2^1 & \dots & V_n^1 \\ O_2 & J_2 & \dots & V_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \dots & J_2 \end{pmatrix},$$

of the form  $T_i = \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix}$  for  $1 \leq i \leq n$ , and  $V_j^i = \begin{pmatrix} v_{2j-1}^{2i-1}(i) & v_{2j}^{2i-1}(i) \\ v_{2j-1}^{2i}(i) & v_{2j}^{2i}(i) \end{pmatrix}$  for  $1 \leq i < j \leq n$ , where each  $t_i$  and  $v_l^k(i)$  is defined by

$$t_i = \frac{a_{[2i]}}{a_{[2i-2]}}, \quad v_l^k(i) = \frac{a_{[2i-2],k,l}}{a_{[2i]}} \quad (2.5)$$

for  $1 \leq i \leq n$  and  $1 \leq k, l \leq 2n$ .

Before we proceed to the proof of the theorem, we illustrate the decomposition by an example. If we take a  $4 \times 4$  skew-symmetric matrix  $A = (a_{ij})_{1 \leq i, j \leq 4}$ , then the above decomposition is given by

$$T = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ -a_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{a_{1234}}{a_{12}} \\ 0 & 0 & -\frac{a_{1234}}{a_{12}} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & \frac{a_{13}}{a_{12}} & \frac{a_{14}}{a_{12}} \\ -1 & 0 & \frac{a_{23}}{a_{12}} & \frac{a_{24}}{a_{12}} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where  $a_{ij} = \text{Pf} \begin{pmatrix} 0 & a_j^i \\ -a_j^i & 0 \end{pmatrix} = a_j^i$  and  $a_{1234} = \text{Pf } A$ .

**Proof of Theorem 2.2.** First we write the matrix  $A$  by  $2 \times 2$  blocks as

$$A = \begin{pmatrix} A_1^1 & A_2^1 & \dots & A_n^1 \\ A_1^2 & A_2^2 & \dots & A_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ A_1^n & A_2^n & \dots & A_n^n \end{pmatrix},$$

where  $A_j^i$  is the  $2 \times 2$  block matrix  $A_j^i = \begin{pmatrix} a_{2j-1}^{2i-1} & a_{2j}^{2i-1} \\ a_{2j-1}^{2i} & a_{2j}^{2i} \end{pmatrix}$  for  $1 \leq i, j \leq n$ . Hence the decomposition (2.4) is equivalent to

$$A_j^i = \sum_{k=1}^{\min(i,j)} {}^t V_i^k T_k V_j^k \quad (2.6)$$

with  $V_i^i = J_2$ . We proceed by induction on  $n$ . If  $n = 1$ , then (2.6) implies  $T = T_1 = A_1^1 = A$  and  $V = V_1^1 = J_2$  so that the existence and uniqueness are trivial. Assume  $n \geq 2$ , and our claim holds for  $n - 1$ . That is, the equations (2.6) for  $1 \leq i, j < n$  uniquely determines all  $T_i$  and  $V_j^i$  for  $1 \leq i, j < n$  and each entry is given by (2.5). This implies that

$$\sum_{k=1}^{i-1} \left( \frac{a_{[2k-1],2i-1} a_{[2k-2],2k,2i}}{a_{[2k-2]} a_{[2k]}} - \frac{a_{[2k-2],2k,2i-1} a_{[2k-1],2i}}{a_{[2k-2]} a_{[2k]}} \right) + \frac{a_{[2i]}}{a_{[2i-2]}} = a_{2i}^{2i-1}$$

holds for  $1 \leq i < n$  from (2.6). Replacing  $(2i - 1)$ st row/column by  $r$ th row/column and  $2i$ th row/column by  $s$ th row/column in this identity, we see

$$\sum_{k=1}^{i-1} \left( \frac{a_{[2k-1],r} a_{[2k-2],2k,s}}{a_{[2k-2]} a_{[2k]}} - \frac{a_{[2k-2],2k,r} a_{[2k-1],s}}{a_{[2k-2]} a_{[2k]}} \right) + \frac{a_{[2i-2],r,s}}{a_{[2i-2]}} = a_s^r \quad (2.7)$$

holds for any  $r$  and  $s$ . From computation of each entry of the equation (2.6), we see that  $v_s^r$  ( $1 \leq i < n$ ,  $r = 2i - 1, 2i$ ,  $s = 2n - 1, 2n$ ) must satisfy

$$\sum_{k=1}^{i-1} \left( \frac{a_{[2k-1],r} a_{[2k-2],2k,s}}{a_{[2k-2]} a_{[2k]}} - \frac{a_{[2k-2],2k,r} a_{[2k-1],s}}{a_{[2k-2]} a_{[2k]}} \right) + \frac{a_{[2i]}}{a_{[2i-2]}} v_s^r = a_s^r.$$

Comparing this equation with (2.7), we see that  $v_s^r$  in (2.5) is the unique solution of this equation. Similarly, from computation of each entry of the equation (2.6),  $t_n$  must satisfy

$$\sum_{k=1}^{n-1} \left( \frac{a_{[2k-1],2n-1} a_{[2k-2],2k,2n}}{a_{[2k-2]} a_{[2k]}} - \frac{a_{[2k-2],2k,2n-1} a_{[2k-1],2n}}{a_{[2k-2]} a_{[2k]}} \right) + t_n = a_{2n}^{2n-1}.$$

Substituting  $i = n - 1$ ,  $r = 2n - 1$  and  $s = 2n$  into (2.7), we obtain

$$\sum_{k=1}^{n-2} \left( \frac{a_{[2k-1],2n-1} a_{[2k-2],2k,2n}}{a_{[2k-2]} a_{[2k]}} - \frac{a_{[2k-2],2k,2n-1} a_{[2k-1],2n}}{a_{[2k-2]} a_{[2k]}} \right) + \frac{a_{[2n-4],2n-1,2n}}{a_{[2n-4]}} = a_{2n}^{2n-1}.$$



Hence we have

$$t_n = \frac{a_{[2n-4],2n-1,2n}}{a_{[2n-4]}} - \frac{a_{[2n-3],2n-1}a_{[2n-4],2n-2,2n}}{a_{[2n-4]}a_{[2n-2]}} + \frac{a_{[2n-4],2n-2,2n-1}a_{[2n-3],2n}}{a_{[2n-4]}a_{[2n-2]}}.$$

Thus, by (2.3), we conclude that  $t_n$  in (2.5) is the unique solution of this equation, and this proves the theorem in the case of  $n$ .  $\square$

This theorem shows that, if we obtain a guess for each entry of  $T$  and  $V$ , then, by uniqueness of the decomposition, it is enough to prove the matrix multiplication, which is equivalent to the single sum

$$\sum_{k \geq 1} \{v_i^{2k-1}(k)t_kv_j^{2k}(k) - v_i^{2k}(k)t_kv_j^{2k-1}(k)\} = a_j^i.$$

From (2.5) it is enough to guess a formula for the subpfaffians  $a_{[2i-1],j}$  and  $a_{[2i-2],2i,j}$  for any row/column indices  $i$  and  $j$ .

By the uniqueness of  $LU$ -decomposition (2.2) and Pfaffian decomposition (2.4), the  $LU$ -decomposition and the Pfaffian decomposition are, in a sense, equivalent. We can get the  $LU$ -decomposition from the Pfaffian decomposition, and vice versa. If we put  $P = (p_j^i)_{i,j \geq 1}$ , where

$$p_j^i = \begin{cases} \delta_j^{i+1} & \text{if } i \text{ is odd,} \\ \delta_j^{i-1} & \text{if } i \text{ is even,} \end{cases}$$

which is the permutation matrix corresponding to  $(12)(34) \dots$ , then it is easy to see that  $\det(PA)_{[i]}^{[i]} \neq 0$  for  $i \geq 1$ . If we put  $J = \bigoplus_{i \geq 1} J_2$ , then  $U = {}^t J V$  is

upper unitriangular,  $D = P T = P {}^t J T J$  is diagonal,  $L = P {}^t V J P$  is lower unitriangular, hence

$$P A = L D U \tag{2.8}$$

gives the  $LU$ -decomposition. Each entry of the matrices  $U = (u_j^i)_{i,j \geq 1}$ ,  $L = (l_j^i)_{i,j \geq 1}$  and  $D = (d_i \delta_j^i)_{i,j \geq 1}$  is given by

$$u_j^i = \begin{cases} -v_j^{i+1} & \text{if } i \text{ is odd,} \\ v_j^{i-1} & \text{if } i \text{ is even,} \end{cases} \quad d_i = \begin{cases} -t_{(i+1)/2} & \text{if } i \text{ is odd,} \\ t_{i/2} & \text{if } i \text{ is even,} \end{cases}$$

$$l_j^i = \begin{cases} v_{i+1}^j & \text{if } i \text{ is odd and } j \text{ is odd,} \\ v_{i-1}^j & \text{if } i \text{ is even and } j \text{ is odd,} \\ -v_{i+1}^j & \text{if } i \text{ is odd and } j \text{ is even,} \\ -v_{i-1}^j & \text{if } i \text{ is even and } j \text{ is even.} \end{cases}$$

For later use we cite the minor summation formula of Pfaffians here:

**Theorem 2.3.** ([14, 15]) Let  $n \leq N$  be positive integers and assume  $n$  is even. Let  $T = (t_j^i)_{1 \leq i \leq n, 1 \leq j \leq N}$  be an  $n \times N$  rectangular matrix, and let  $B = (b_j^i)_{1 \leq i, j \leq N}$  be a skew symmetric matrix of size  $N$ . Then we have

$$\sum_{\substack{I \subseteq [N] \\ \#I = n}} \text{Pf}(B_I) \det(T_I^{[n]}) = \text{Pf}(Q), \quad (2.9)$$

where the skew symmetric matrix  $Q = (Q_j^i) = TB^tT$  of size  $n$  whose entries are given by

$$Q_j^i = \sum_{1 \leq k < l \leq N} b_l^k \det(T_{kl}^{ij}), \quad (1 \leq i, j \leq n). \quad (2.10)$$

When  $n$  is odd, we can immediately derive a similar formula from the case when  $n$  is even.

**Proposition 2.4.** Let  $\{\alpha_k\}_{k \geq 1}$  be any sequence, and let  $n$  be a positive integer. Set  $B = (b_j^i)_{i, j \geq 1}$  to be the skew-symmetric matrix defined by

$$b_j^i = \begin{cases} \alpha_i & \text{if } j = i + 1 \text{ for } i \geq 1, \\ -\alpha_j & \text{if } i = j + 1 \text{ for } j \geq 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.11)$$

If  $I = (i_1, \dots, i_{2n})$  is an index set such that  $1 \leq i_1 < \dots < i_{2n}$ , then

$$\text{Pf}(B_I) = \begin{cases} \prod_{k=1}^n \alpha_{i_{2k-1}} & \text{if } i_{2k} = i_{2k-1} + 1 \text{ for } k = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases} \quad (2.12)$$

### 3 A Pfaffian analogue of $q$ -Catalan Hankel determinants

Let us write

$$a_j^i = (q^{i-1} - q^{j-1}) \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} \quad (3.1)$$

for  $i, j \geq 1$ , and let  $A$  denote the skew-symmetric matrix

$$A = (a_j^i)_{i, j \geq 1}$$

of infinite degree. Then the following theorem gives the Pfaffian decomposition of  $A$ .

**Theorem 3.1.** Let  $A$  be as above, and let

$$t_i = a^{i-1} q^{(i-1)(i+r)} \frac{(q; q)_i (aq; q)_{i+r} (bq; q)_{i-1}}{(abq^2; q)_{2i+r-1} (abq^{i+r}; q)_{i-1}},$$

$$v_j^i = \begin{cases} o_j^i & \text{if } i \text{ is odd,} \\ e_j^i & \text{if } i \text{ is even,} \end{cases}$$

where

$$o_j^i = \frac{(q^{j-i}; q)_i}{(q; q)_i} \cdot \frac{(aq^{i+r+1}; q)_{j-i-1}}{(abq^{2i+r+1}; q)_{j-i-1}},$$

$$e_j^i = q \frac{(q^{j-i}; q)_1 (q^{j-i+2}; q)_{i-2}}{(q; q)_{i-1}} \cdot \frac{(aq^{i+r}; q)_{j-i} f(i, j, r)}{(abq^{2i+r-3}; q)_1 (abq^{2i+r-1}; q)_{j-i+1}},$$

with

$$f(i, j, r) = (1 - q^{i-1})(1 - aq^{i+r-1})(1 - abq^{i+j+r-2})/(1 - q) + aq^{2i+r-3}(1 - b)(1 - q^{j-i+1}). \quad (3.2)$$

If we put  $T = \bigoplus_{i \geq 1} \begin{pmatrix} 0 & t_{2i-1} \\ -t_{2i-1} & 0 \end{pmatrix}$  and  $V = (v_j^i)_{i,j \geq 1}$  then

$$A = {}^t V T V \quad (3.3)$$

gives the Pfaffian decomposition of  $A$ .

An immediate consequence of the theorem is the following corollary.

**Corollary 3.2.** Let  $n \geq 1$  and  $r$  be integers. Then we have

$$\begin{aligned} & \text{Pf} \left( (q^{i-1} - q^{j-1}) \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} \right)_{1 \leq i, j \leq 2n} \\ &= a^{n(n-1)} q^{n(n-1)(4n+1)/3 + n(n-1)r} \prod_{k=1}^{n-1} (bq; q)_{2k} \prod_{k=1}^n \frac{(q; q)_{2k-1} (aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}}. \end{aligned} \quad (3.4)$$

In fact, we obtain a more general formula from Theorem 3.1. If  $A$  is as above and  $m$  is a positive integer, then the following identities hold:

$$\begin{aligned} \text{Pf} \left( A_{[2n-1], m} \right) &= a^{n(n-1)} q^{n(n-1)(4n+1)/3+n(n-1)r} \\ &\times \frac{(q^{m-2n+1}; q)_{2n-1} (aq; q)_{m+r-1}}{(abq^2; q)_{m+2n+r-3}} \prod_{k=1}^{n-1} \frac{(bq; q)_{2k} (q; q)_{2k-1} (aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}}, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \text{Pf} \left( A_{[2n-2], 2n, m} \right) &= a^{n(n-1)} q^{n(n-1)(4n+1)/3+n(n-1)r+1} f(2n, m, r) \\ &\times \frac{(q^{m-2n}; q)_1 (q^{m-2n+2}; q)_{2n-2} (aq; q)_{m+r-1}}{(abq^{4n+r-3}; q)_1 (abq^2; q)_{m+2n+r-2}} \prod_{k=1}^{n-1} \frac{(bq; q)_{2k} (q; q)_{2k-1} (aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}}, \end{aligned} \quad (3.6)$$

where  $f(i, j, r)$  is defined by (3.2).

Next we consider a specialization of Corollary 3.2. If we put  $a = q^\alpha$  and  $b = q^\beta$  and let  $q \rightarrow 1$  in (3.4), then we obtain the following corollary:

**Corollary 3.3.** Let  $n \geq 1$  and  $r$  be integers. Then we have

$$\begin{aligned} \text{Pf} \left( (j-i) \frac{(\alpha+1)_{i+j+r-2}}{(\alpha+\beta+2)_{i+j+r-2}} \right)_{1 \leq i, j \leq 2n} \\ = \prod_{k=1}^{n-1} (\beta+1)_{2k} \prod_{k=1}^n \frac{(2k-1)! (\alpha+1)_{2k+r-1}}{(\alpha+\beta+2)_{2(k+n)+r-3}}, \end{aligned} \quad (3.7)$$

where we use the notation

$$(\alpha)_n = \begin{cases} \prod_{i=1}^n (\alpha+i-1) & \text{if } n \geq 0, \\ 1 / \prod_{i=1}^{-n} (\alpha+i+n-1) & \text{if } n < 0. \end{cases}$$

An almost equivalent result is obtained in [18, Theorem 6], which is motivated by work in [5, 22]. In [8] Ciucu and Krattenthaler use a special case of this Pfaffian for application to certain exact enumeration of lozenge tiling. Further, if we put  $\alpha = -\frac{1}{2}$  and  $\beta = \frac{1}{2}$  in (3.7), then we obtain

$$\begin{aligned} \text{Pf} \left( (j-i) C_{i+j+r-2} \right)_{1 \leq i, j \leq 2n} \\ = \prod_{k=1}^{n-1} \frac{(4k+1)!}{(2k)!} \prod_{k=1}^n \frac{(2k-1)! (4k+2r-2)!}{(2k+r-1)! \{2(k+n)+r-2\}!}, \end{aligned} \quad (3.8)$$

where  $C_n = \frac{1}{n+1} \binom{2n}{n}$  denotes the *Catalan numbers*. On the other hand, if we put  $\alpha = -\frac{1}{2}$  and  $\beta = -\frac{1}{2}$  in (3.7), then we obtain

$$\begin{aligned} & \text{Pf} \left( (j-i) C_{i+j+r-2}^{(D)} \right)_{1 \leq i, j \leq 2n} \\ &= \prod_{k=1}^{n-1} \frac{(4k)!}{(2k)!} \prod_{k=1}^n \frac{(2k-1)!(4k+2r-2)!}{(2k+r-1)!\{2(k+n)+r-3\}!}, \end{aligned} \quad (3.9)$$

where  $C_n^{(D)} = \binom{2n}{n}$  is usually called the *central binomial coefficients*.

The *Laguerre polynomials* (see [19]) are defined by

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1 \left( \begin{matrix} -n \\ \alpha+1 \end{matrix}; x \right),$$

which are orthogonal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-x} x^\alpha f(x) g(x) dx.$$

Note that

$$\text{Pf}(c_i c_j a_j^i)_{1 \leq i, j \leq 2n} = c_1 \dots c_{2n} \text{Pf}(a_j^i)_{1 \leq i, j \leq 2n}. \quad (3.10)$$

Multiplying (3.7) by  $\beta^{n(2n+1)+n(r-2)}$  and then letting  $\beta \rightarrow \infty$  we get the following result.

**Corollary 3.4.** Let  $\mu_n = (\alpha+1)_n$  for  $n \geq 0$ , which is known to be the moment sequence of Laguerre polynomials. Then we have

$$\text{Pf} \left( (j-i) \mu_{i+j+r-2} \right)_{1 \leq i, j \leq 2n} = \prod_{k=1}^n (2k-1)! (\alpha+1)_{2k+r-1}. \quad (3.11)$$

The *Hermite polynomials* (see [19]) are defined by

$$H_n(x) = (2x)^n {}_2F_0 \left( \begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix}; -\frac{1}{x^2} \right),$$

which are orthogonal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-x^2} f(x) g(x) dx.$$

Substituting  $\alpha = \frac{1}{2}$  in (3.11) we get another remarkable formula.

**Corollary 3.5.** Let  $\mu_n = \prod_{k=0}^n (2k+1)$  denote the double factorial of  $2n+1$  for  $n \geq 0$ , which is known to be the moment sequence of Hermite polynomials. Then we have

$$\text{Pf} \left( (j-i) \mu_{i+j+r-2} \right)_{1 \leq i, j \leq 2n} = \frac{1}{2^n} \prod_{k=1}^n (4k-2)!! (4k+2r-1)!! \quad (3.12)$$

## 4 Proof of Theorem 3.1

Let  $a_j^i$ ,  $t_i$  and  $v_j^i$  be as in Theorem 3.1. To prove (3.3), it is enough to show

$$\sum_{k \geq 1} (v_i^{2k-1} t_{2k-1} v_j^{2k} - v_i^{2k} t_{2k-1} v_j^{2k-1}) = a_j^i \quad (4.1)$$

for  $i, j \geq 1$ . Replacing  $aq^r$  by  $a$ , we may assume  $r = 0$  hereafter without loss of generality. Hence (4.1) is written as

$$\begin{aligned} & \sum_{k \geq 1} a^{2(k-1)} q^{2(k-1)(2k-1)+1} \cdot \frac{(q^{i-2k+2}; q)_{2k-2} (q^{j-2k+2}; q)_{2k-2}}{(q; q)_{2k-1}} \\ & \cdot \frac{(aq^{2k}; q)_{i-2k} (aq; q)_{j-1} (bq; q)_{2(k-1)}}{(abq^{2k-1}; q)_{2k-1} (abq^{4k-1}; q)_{i-2k+1} (abq^2; q)_{j+2k-2}} \\ & \cdot \left\{ (1 - q^{i-2k+1})(1 - q^{j-2k})(1 - abq^{i+2k-1}) f(2k, j, 0) \right. \\ & \quad \left. - (1 - q^{i-2k})(1 - q^{j-2k+1})(1 - abq^{j+2k-1}) f(2k, i, 0) \right\} = a_j^i. \end{aligned}$$

Replacing  $2k-1$  by  $k$ , we obtain

$$\begin{aligned} & \sum_{\substack{k \geq 1 \\ k \text{ odd}}} a^{k-1} q^{k(k-1)+1} \cdot \frac{(abq^{2k}; q)_1 (abq^2; q)_{k-2} (bq, q^{i-k+1}, q^{j-k+1}; q)_{k-1}}{(q, aq, abq^{i+1}, abq^{j+1}; q)_k} \\ & \times g_k(i, j; a, b, q) = \frac{(aq; q)_{i+j-2} (abq^2; q)_{i-1} (abq^2; q)_{j-1}}{(aq; q)_{i-1} (aq; q)_{j-1} (abq^2; q)_{i+j-2}}, \quad (4.2) \end{aligned}$$

where  $g_k(i, j; a, b, q)$  is set to be

$$\begin{aligned} g_k(i, j; a, b, q) &= (1 - q^k)(1 - aq^k) \\ & \times \left\{ q^{-k} (1 + abq^{2k}) (1 + abq^{i+j-1}) - ab(1 + q) (q^{i-1} + q^{j-1}) \right\} \\ & + aq^{k-1} (1 - b) (1 - q^{i-k}) (1 - q^{j-k}) (1 - abq^{2k+1}). \quad (4.3) \end{aligned}$$

By numeric experiments we observe that (4.2) also holds in the case where the sum in the left-hand side runs over all nonnegative even integers  $k$ , i.e.,

$$\sum_{\substack{k \geq 0 \\ k \text{ even}}} a^{k-1} q^{k(k-1)+1} \cdot \frac{(abq^{2k}; q)_1 (abq^2; q)_{k-2} (bq, q^{i-k+1}, q^{j-k+1}; q)_{k-1}}{(q, aq, abq^{i+1}, abq^{j+1}; q)_k} \\ \times g_k(i, j; a, b, q) = \frac{(aq; q)_{i+j-2} (abq^2; q)_{i-1} (abq^2; q)_{j-1}}{(aq; q)_{i-1} (aq; q)_{j-1} (abq^2; q)_{i+j-2}}. \quad (4.4)$$

By adding or subtracting (4.2) and (4.4), these two identities are equivalent to

$$\sum_{k \geq 0} a^{k-1} q^{k(k-1)+1} \cdot \frac{(abq^{2k}; q)_1 (abq^2; q)_{k-2} (bq, q^{i-k+1}, q^{j-k+1}; q)_{k-1}}{(q, aq, abq^{i+1}, abq^{j+1}; q)_k} \\ \times g_k(i, j; a, b, q) = \frac{2(aq; q)_{i+j-2} (abq^2; q)_{i-1} (abq^2; q)_{j-1}}{(aq; q)_{i-1} (aq; q)_{j-1} (abq^2; q)_{i+j-2}}, \quad (4.5)$$

and

$$\sum_{k \geq 0} (-1)^k a^{k-1} q^{k(k-1)+1} \cdot \frac{(abq^{2k}; q)_1 (abq^2; q)_{k-2} (bq, q^{i-k+1}, q^{j-k+1}; q)_{k-1}}{(q, aq, abq^{i+1}, abq^{j+1}; q)_k} \\ \times g_k(i, j; a, b, q) = 0. \quad (4.6)$$

To prove (4.5) we rewrite  $g_k(i, j; a, b, q)$  as follows and apply  $q$ -Dougall formula, i.e., Lemma 4.1, to each term, then a direct computation leads to the desired identity:

$$g_k(i, j; a, b, q) = q^{-1-k} (q + aq^{i+j}) (1 - q^k) (1 - q^{k-1}) (1 - abq^k) (1 - abq^{k+1}) \\ + q^{-1} \left\{ a(bq - ab - 1 + b) (1 - q^{i+j}) + (1 - a)(q - ab) \right. \\ \left. + a(1 + bq) (1 - q^i) (1 - q^j) + (1 + aq^{i+j-1}) (1 - q) (1 - abq) \right\} (1 - q^k) (1 - abq^k) \\ + aq^{k-1} (1 - b) (1 - abq) (1 - q^i) (1 - q^j).$$

To prove (4.6), we generalize this identity as

$$\sum_{k=0}^m (-1)^k a^{k-1} q^{k(k-1)+1} \\ \times \frac{(1 - abq^{2k}) (abq^2; q)_{k-2} (bq, cq^{-k+1}, dq^{-k+1}; q)_{k-1} \widehat{g}_k(a, b, c, d, q)}{(q, aq, abcq, abdq; q)_k} \\ = \frac{a^m c^m d^m (1 - abq^{2m+1}) (abq^2; q)_{m-1} (bq, q/c, q/d; q)_m}{(-q)^m (q, aq, abcq, abdq; q)_m}. \quad (4.7)$$

where

$$\begin{aligned} \widehat{g}_k(a, b, c, d, q) &= (1 - q^k)(1 - aq^k) \\ &\times \left\{ q^{-k}(1 + abq^{2k})(1 + abcdq^{-1}) - abq^{-1}(1 + q)(c + d) \right\} \\ &+ aq^{k-1}(1 - b)(1 - cq^{-k})(1 - dq^{-k})(1 - abq^{2k+1}). \end{aligned} \quad (4.8)$$

Then (4.7) is proven by induction on  $m$ . This completes the proof of Theorem 3.1.

**Lemma 4.1.** Let  $m$  be an integer. Then we have

$$\begin{aligned} &\sum_{k \geq m} a^{k-m} q^{k(k-m)} \cdot \frac{(1 - abq^{2k})(abq^2; q)_{k+m-2}(bq, q^{i-k+1}, q^{j-k+1}; q)_{k-1}}{(q; q)_{k-m}(aq, abq^{i+1}, abq^{j+1}; q)_k} \\ &= (q^{i-m+1}, q^{j-m+1}, bq; q)_{m-1} \cdot \frac{(aq^{j+1}; q)_{i-m}(abq^2; q)_{i-1}}{(aq; q)_i(abq^{j+1}; q)_i}. \end{aligned} \quad (4.9)$$

In fact (4.9) reduces to the  $q$ -Dougall formula (Jackson's formula) [4, (12.3.2)], [9, (2.4.2)]

$${}_6\phi_5 \left[ \begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix}; q, \frac{aq^{n+1}}{bc} \right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}, \quad (4.10)$$

by the substitution

$$a \leftarrow abq^{2m}, \quad b \leftarrow bq^m, \quad c \leftarrow q^{m-j}, \quad n \leftarrow i - m.$$

**Remark 4.2.** The lemma also directly follows from the Bailey pair  $(\alpha_n, \beta_n)$  given by

$$\begin{aligned} \alpha_n &= \frac{(a, b, c; q)_n(1 - aq^{2n})(aq/bc)^n(-1)^n q^{\binom{n}{2}}}{(q, aq/b, aq/c; q)_n(1 - a)}, \\ \beta_n &= \frac{(aq/bc; q)_n}{(q, aq/b, aq/c; q)_n}. \end{aligned}$$

Here a pair  $(\alpha_n, \beta_n)$  is said to be a *Bailey pair* [4] if it satisfies

$$\beta_n = \sum_{k=0}^n \frac{\alpha_k}{(q; q)_{n-k}(aq; q)_{n+k}}.$$



In fact we prove two identities (4.2) and (4.4) in this section. While (4.2) is used to prove Theorem 3.1, one may ask what's (4.4) for? In fact we can interpret (4.4) as a Pfaffian decomposition of another skew-symmetric matrix. Define  $a_j^i$  for  $i, j \geq 0$  by

$$a_j^0 = \frac{(abq^{r-1}; q)_1 (aq; q)_{j+r-1}}{aq^r (1-b) (abq^2; q)_{j+r-2}}, \quad (4.11)$$

with  $a_0^i = -a_i^0$ ,  $a_0^0 = 0$  and (3.1) for  $i, j \geq 1$ .

**Theorem 4.3.** Let  $\tilde{A} = (a_j^i)_{i,j \geq 0}$  where  $a_j^i$  is as above. Let  $t_i$ ,  $o_j^i$  and  $e_j^i$  be as in Theorem 3.1, and we put

$$\tilde{v}_j^i = \begin{cases} o_j^i & \text{if } i \text{ is even,} \\ e_j^i & \text{if } i \text{ is odd.} \end{cases}$$

If we set  $\tilde{T} = \bigoplus_{i \geq 0} \begin{pmatrix} 0 & t_{2i} \\ -t_{2i} & 0 \end{pmatrix}$  and  $\tilde{V} = (\tilde{v}_j^i)_{i,j \geq 0}$  then

$$\tilde{A} = t \tilde{V} \tilde{T} \tilde{V} \quad (4.12)$$

gives the Pfaffian decomposition of  $\tilde{A}$ .

**Corollary 4.4.** Let  $n \geq 1$  and  $r$  be integers. Then we have

$$\begin{aligned} \text{Pf} (a_j^i)_{0 \leq i, j \leq 2n-1} &= a^{n(n-2)} q^{n(n-1)(4n-5)/3 + n(n-2)r} \\ &\times \prod_{k=0}^{n-1} \frac{(q; q)_{2k} (aq; q)_{2k+r} (bq; q)_{2k-1}}{(abq^2; q)_{4k+r-1} (abq^{2k+r}; q)_{2k-1}}. \end{aligned} \quad (4.13)$$

Let  $P_{n,r}(a, b; q)$  denote the right-hand side of (4.13). Then, more generally we have

$$\text{Pf} \left( \tilde{A}_{[0, 2n-2], m-1} \right) = \frac{(q^{m-2n+1}; q)_{2n-2} (aq^{2n+r-1}; q)_{m-2n}}{(q; q)_{2n-2} (abq^{4n+r-3}; q)_{m-2n}} P_{n,r}(a, b; q), \quad (4.14)$$

$$\begin{aligned} \text{Pf} \left( \tilde{A}_{[0, 2n-3], 2n-1, m-1} \right) &= q \cdot \frac{(q^{m-2n}; q)_1 (q^{m-2n+2}; q)_{2n-3} (aq^{2n+r-1}; q)_{m-2n}}{(q; q)_{2n-2} (abq^{4n+r-5}; q)_1 (abq^{4n+r-3}; q)_{m-2n+1}} \\ &\times f(2n-1, m-1, r) P_{n,r}(a, b; q). \end{aligned} \quad (4.15)$$

Here we use the notation  $[i, j] = \{x \in \mathbb{Z} \mid i \leq x \leq j\}$ .

## 5 Weighted enumeration of shifted RPPs

In this section we give an application of Corollary 3.2, which enumerates a certain class of shifted reverse plane partitions.

**Definition 5.1.** A *shifted reverse plane partition* (abbreviated as *shifted RPP*) is an array  $\pi = (\pi_{ij})$  of nonnegative integers, defined only for  $i \leq j$ , that has nondecreasing rows and columns, and that can be written in the form

$$\begin{array}{ccccccc} \pi_{11} & \pi_{12} & \pi_{13} & \cdots & \cdots & \cdots & \pi_{1,\lambda_1} \\ & \pi_{22} & \pi_{23} & \cdots & \cdots & \pi_{2,\lambda_2+1} & \\ & & \ddots & \vdots & \vdots & \ddots & \\ & & & \pi_{n,n} & \cdots & \pi_{n,\lambda_n+n-1} & \end{array}, \quad (5.1)$$

where

- (i)  $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ ,
- (ii)  $\pi_{i,j} \leq \pi_{i,j+1}$  whenever the both sides are defined,
- (iii)  $\pi_{i,j} \leq \pi_{i+1,j}$  whenever the both sides are defined.

Further, if  $\pi$  also satisfies

- (iii')  $\pi_{i,j} < \pi_{i+1,j}$  whenever the both sides are defined,

then it is called *column-strict* shifted reverse plane partition or a *shifted tableau*. The entries  $\pi_{ij}$  are called the *parts* of  $\pi$ . To each shifted reverse plane partition  $\pi$  we assign the *weight*  $|\pi| = \sum_{ij} \pi_{ij}$  to be the sum of parts. The strict partition  $\lambda$  is called the *shape* of  $\pi$ , and the nondecreasing sequence  $(\pi_{11}, \pi_{22}, \dots, \pi_{nn})$  is called the *profile* of  $\pi$ . Let  $\mathcal{R}_{\lambda,\mu}$  denote the set of all shifted reverse plane partitions of shape  $\lambda$  and profile  $\mu$ , and  $\mathcal{T}_{\lambda,\mu}$  the set of all shifted tableaux of shape  $\lambda$  and profile  $\mu$  for fixed  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$  with  $\lambda_1 > \cdots > \lambda_n > 0$  and  $0 \leq \mu_1 \leq \cdots \leq \mu_n$ .

For example,

0	0	0	0	1	1	1	2	2	3	4
	1	1	2	2	2	3	3	4		
		3	3	3	4	4	5			
			5	5	5					

is a shifted tableau of shape  $\lambda = (11, 8, 6, 3)$  and profile  $\mu = (0, 1, 3, 5)$  with weight 69. If  $\mathcal{F}$  is a family of shifted reverse plane partitions, then the generating function of  $\mathcal{F}$  is defined to be

$$\text{GF}[\mathcal{F}] = \sum_{\pi \in \mathcal{F}} q^{|\pi|}. \quad (5.2)$$

Let  $n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$ . It is easy to see that

$$\text{GF}[\mathcal{T}_{\lambda, \nu + \epsilon_n}] = q^{n(\lambda)} \text{GF}[\mathcal{R}_{\lambda, \nu}], \quad (5.3)$$

where  $\epsilon_n = (0, 1, \dots, n-1)$  and  $\nu = (\nu_1, \nu_2, \dots, \nu_n)$  is a profile such that  $\nu_1 \leq \nu_2 \leq \dots \leq \nu_n$ . Let  $\mathcal{P}_n$  denote the set of profiles  $\nu = (\nu_1, \dots, \nu_{2n})$  such that  $0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_{2n}$  and  $\nu_{2k} = \nu_{2k-1}$  for  $k = 1, \dots, n$ . For  $\nu \in \mathcal{P}_n$  and  $x \in \mathbb{Z}$  we let

$$\omega_x(\nu) = (aq^x)^{|\nu|/2} \prod_{k=1}^n \frac{(bq^{2k-1}; q)_{\nu_{2k-1}}}{(q^{2k-1}; q)_{\nu_{2k-1}}}, \quad (5.4)$$

where  $|\nu| = \sum_{k=1}^{2n} \nu_k$ . Let  $\mathcal{P}'_n = \{\nu + \epsilon_{2n} \mid \nu \in \mathcal{P}_n\}$ , and

$$\omega'_x(\mu) = (aq^x)^{(|\mu|-n)/2} \prod_{k=1}^n \frac{(bq; q)_{\mu_{2k-1}}}{(q; q)_{\mu_{2k-1}}} \quad (5.5)$$

for  $\mu \in \mathcal{P}'_n$ . Now we are in position to state our main theorem in this section. If the shape  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$  is in the form of  $\lambda = (m, m-1, \dots, m-r+1)$  for positive integers  $m \geq r$ , then it is called *staircase*. Each of (3.4), (3.5)

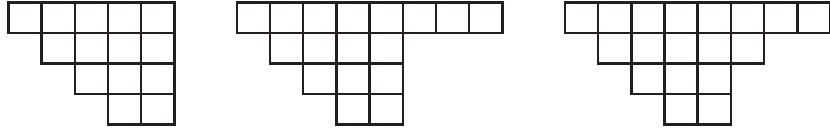


Figure 1: Nearly Staircase Shapes

and (3.6) corresponds to each of (5.6), (5.7) and (5.8) below. In fact the leftmost diagram in Figure 1 gives the case of  $m = 5$ ,  $n = 2$  in (5.6), and the middle (resp. rightmost) diagram in Figure 1 gives the case of  $l = 8$ ,  $m = 5$ ,  $n = 2$  in (5.7) (resp. (5.8)).

**Theorem 5.2.** Let  $r$  be an integer. For any positive integers  $m$  and  $n$  such that  $m \geq 2n$ , we fix the shape  $\lambda = (m, m-1, \dots, m-2n+1)$  of length  $2n$ . Then we have

$$\begin{aligned} & \sum_{\nu \in \mathcal{P}_n} \omega_{r-2(m-2n)-1}(\nu) \text{GF}[\mathcal{R}_{\lambda, \nu}] \\ &= \left\{ \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \right\}^n \cdot \prod_{k=1}^{2n} \frac{(q; q)_{k-1}}{(q; q)_{k+m-2n-1}} \cdot \prod_{k=1}^n \frac{(aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}}. \end{aligned} \quad (5.6)$$

More generally, if  $\lambda = (l, m-1, m-2, m-3, \dots, m-2n+1)$  where  $l \geq m$ , then we have

$$\begin{aligned} & \sum_{\nu \in \mathcal{P}_n} \omega_{r-2(m-2n)-1}(\nu) \text{GF}[\mathcal{R}_{\lambda, \nu}] = \left\{ \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \right\}^n \cdot \frac{(q^{l-m+1}; q)_{2n-1}}{(q; q)_{l-1}} \\ & \times \prod_{k=1}^{2n-1} \frac{(q; q)_{k-1}}{(q; q)_{k+m-2n-1}} \cdot \frac{(aq; q)_{l-m+2n+r-1}}{(abq^2; q)_{l-m+4n+r-3}} \cdot \prod_{k=1}^{n-1} \frac{(aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}}, \end{aligned} \quad (5.7)$$

and if  $\lambda = (l, m, m-2, m-3, \dots, m-2n+1)$  where  $l > m$ , then we have

$$\begin{aligned} & \sum_{\nu \in \mathcal{P}_n} \omega_{r-2(m-2n)-1}(\nu) \text{GF}[\mathcal{R}_{\lambda, \nu}] \\ &= \left\{ \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \right\}^n \cdot \frac{(q^{l-m}; q)_1 (q^{l-m+2}; q)_{2n-2}}{(q; q)_{l-1} (q; q)_{m-1}} \cdot \frac{\prod_{k=1}^{2n-1} (q; q)_{k-1}}{\prod_{k=1}^{2n-2} (q; q)_{k+m-2n-1}} \\ & \times \frac{(aq; q)_{l-m+2n+r-1} f(2n, l-m+2n, r)}{(abq^{4n+r-3}; q)_1 (abq^2; q)_{l-m+4n+r-2}} \prod_{k=1}^{n-1} \frac{(aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}}. \end{aligned} \quad (5.8)$$

To prove this theorem, we first recall the notation of the lattice path method, which is due to Gessel and Viennot [10]. Let  $D = (V, E)$  be an acyclic digraph without multiple edges. If  $u$  and  $v$  are any pair of vertices, let  $\mathcal{P}(u, v)$  denote the set of all directed paths from  $u$  to  $v$ . For a fixed positive integer  $n$ , an  $n$ -vertex is an  $n$ -tuple of vertices of  $D$ . If  $\mathbf{u} = (u_1, \dots, u_n)$  and  $\mathbf{v} = (v_1, \dots, v_n)$  are  $n$ -vertices, an  $n$ -path from  $\mathbf{u}$  to  $\mathbf{v}$  is an  $n$ -tuple  $\mathbf{P} = (P_1, \dots, P_n)$  such that  $P_i \in \mathcal{P}(u_i, v_i)$ ,  $i = 1, \dots, n$ . The  $n$ -path  $\mathbf{P} = (P_1, \dots, P_n)$  is said to be *non-intersecting* if any two different paths  $P_i$  and  $P_j$  have no vertex in common. We will write  $\mathcal{P}(\mathbf{u}, \mathbf{v})$  for the set of all  $n$ -paths from  $\mathbf{u}$  to  $\mathbf{v}$ , and write  $\mathcal{P}_0(\mathbf{u}, \mathbf{v})$  for the subset of  $\mathcal{P}(\mathbf{u}, \mathbf{v})$  consisting of non-intersecting  $n$ -paths. If  $\mathbf{u} = (u_1, \dots, u_m)$  and  $\mathbf{v} = (v_1, \dots, v_n)$  are linearly

ordered sets of vertices of  $D$ , then  $\mathbf{u}$  is said to be  $D$ -compatible with  $\mathbf{v}$  if every path  $P \in \mathcal{P}(u_i, v_l)$  intersects with every path  $Q \in \mathcal{P}(u_j, v_k)$  whenever  $i < j$  and  $k < l$ . If  $\pi \in S_n$ , by  $\mathbf{v}^\pi$  we mean the  $n$ -vertex  $(v_{\pi(1)}, \dots, v_{\pi(n)})$ . The weight  $w(\mathbf{P})$  of an  $n$ -path  $\mathbf{P}$  is defined to be the product of the weights of its components. Thus, if  $\mathbf{u} = (u_1, \dots, u_n)$  and  $\mathbf{v} = (v_1, \dots, v_n)$  are  $n$ -vertices, we define the generating functions  $F(\mathbf{u}, \mathbf{v}) = \text{GF}[\mathcal{P}(\mathbf{u}, \mathbf{v})] = \sum_{\mathbf{P} \in \mathcal{P}(\mathbf{u}, \mathbf{v})} w(\mathbf{P})$  and  $F_0(\mathbf{u}, \mathbf{v}) = \text{GF}[\mathcal{P}_0(\mathbf{u}, \mathbf{v})] = \sum_{\mathbf{P} \in \mathcal{P}_0(\mathbf{u}, \mathbf{v})} w(\mathbf{P})$ . In particular, if  $u$  and  $v$  are any pair of vertices, we write

$$h(u, v) = \text{GF}[\mathcal{P}(u, v)] = \sum_{P \in \mathcal{P}(u, v)} w(P).$$

**Lemma 5.3.** (Lindström-Gessel-Viennot [10])

Let  $\mathbf{u} = (u_1, \dots, u_n)$  and  $\mathbf{v} = (v_1, \dots, v_n)$  be two  $n$ -vertices in an acyclic digraph  $D$ . Then

$$\sum_{\pi \in S_n} \text{sgn } \pi F_0(\mathbf{u}^\pi, \mathbf{v}) = \det[h(u_i, v_j)]_{1 \leq i, j \leq n}. \quad (5.9)$$

In particular, if  $\mathbf{u}$  is  $D$ -compatible with  $\mathbf{v}$ , then

$$F_0(\mathbf{u}, \mathbf{v}) = \det[h(u_i, v_j)]_{1 \leq i, j \leq n}. \quad (5.10)$$

Using the Lindström-Gessel-Viennot theorem, we obtain the following determinantal expression for the generating function of shifted tableaux.

**Lemma 5.4.** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$  be sequences such that  $\lambda_1 > \dots > \lambda_n > 0$  and  $0 \leq \mu_1 < \dots < \mu_n$ . Then

$$\text{GF}[\mathcal{T}_{\lambda, \mu}] = \det \left( \frac{q^{\lambda_i \mu_j}}{(q; q)_{\lambda_i - 1}} \right)_{1 \leq i, j \leq n}. \quad (5.11)$$

**Proof.** We consider the digraph  $D$  whose vertex set is  $\mathbb{Z}_{\geq 0}^2$  and the edge set is defined as follows. An edge is directed from  $u$  to  $v$  whenever  $v - u = (1, 0)$  or  $(0, 1)$  (resp. whenever  $v - u = (1, 0)$ ) if the vertex  $u = (i, j)$  satisfies  $i > 0$  (resp.  $i = 0$ ). For  $u = (i, j)$ , we assign the weight  $q^j$  (resp. 1) to the edge with  $v - u = (1, 0)$  (resp.  $(0, 1)$ ). Fix a sufficiently large positive integer  $N$ . For given  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $\mu = (\mu_1, \dots, \mu_n)$ , we set the vertices  $u_i = (0, \mu_i)$  and  $v_j = (\lambda_j, N)$  for  $i, j = 1, \dots, n$ . Let  $\mathcal{T}_{\lambda, \mu}^N$  denote the set of shifted tableaux  $\pi \in \mathcal{T}_{\lambda, \mu}$  such that each part is less than or equal to  $N$ .

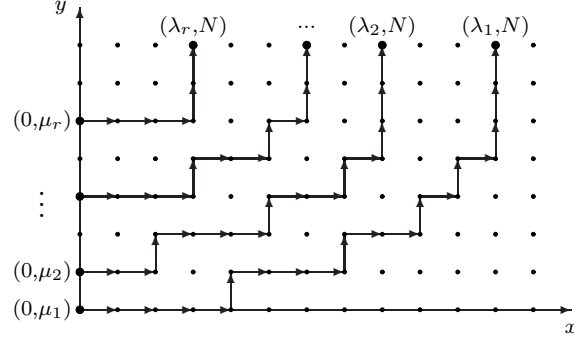


Figure 2: The  $n$ -path corresponding to the above shifted tableaux

Then a tableau  $\pi \in \mathcal{T}_{\lambda, \mu}^N$  is interpreted as an  $n$ -path  $\mathbf{P}$  from  $\mathbf{u} = (u_1, \dots, u_n)$  to  $\mathbf{v} = (v_1, \dots, v_n)$ . For instance the shifted tableaux in the above example is pictorially illustrated by the 4-path in Figure 2. If  $u = (0, y)$  and  $v = (x, N)$ , then we have  $h(u, v) = q^{xy} \left[ \begin{smallmatrix} x-1+N-y \\ x-1 \end{smallmatrix} \right]_q$ . Hence, by Lemma 5.3, we obtain

$$\text{GF} [\mathcal{T}_{\lambda, \mu}^N] = \det \left( q^{\lambda_i \mu_j} \left[ \begin{smallmatrix} \lambda_i - 1 + N - \mu_j \\ \lambda_i - 1 \end{smallmatrix} \right]_q \right)_{1 \leq i, j \leq n}.$$

Letting  $N \rightarrow \infty$ , we obtain the desired identity (5.11).  $\square$

**Proof of Theorem 5.2.** In fact (5.6) is equivalent to the following identity from (5.3):

$$\begin{aligned} & \sum_{\mu \in \mathcal{P}'_n} \omega'_{r-2(m-2n)-1}(\mu) \text{GF} [\mathcal{T}_{\lambda, \mu}] \\ &= a^{n(n-1)} q^{mn+n(n-2)(4n-1)/3+n(n-1)r} \left\{ \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \right\}^n \\ & \times \prod_{k=1}^{2n} \frac{1}{(q; q)_{k+m-2n-1}} \prod_{k=1}^{n-1} (bq; q)_{2k} \prod_{k=1}^n \frac{(q; q)_{2k-1} (aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}}. \end{aligned} \quad (5.12)$$

Now the entries of the skew-symmetric matrix in Corollary 3.2 can be written as

$$a_j^i = (q^{i-1} - q^{j-1}) \frac{(aq; q)_{\infty}}{(abq^2; q)_{\infty}} \cdot \frac{(abq^{i+j+r}; q)_{\infty}}{(aq^{i+j+r-1}; q)_{\infty}}.$$

If we apply the  $q$ -binomial theorem [9, (1.3.2)]

$$\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k = \frac{(ax; q)_{\infty}}{(x; q)_{\infty}}, \quad (5.13)$$

then we obtain

$$\begin{aligned} a_j^i &= (q^{i-1} - q^{j-1}) \frac{(aq; q)_{\infty}}{(abq^2; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(bq; q)_k}{(q; q)_k} (aq^{i+j+r-1})^k \\ &= q^{-s} \frac{(aq; q)_{\infty}}{(abq^2; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(bq; q)_k}{(q; q)_k} (aq^{r-2s+1})^k \left| \frac{q^{(i+s-1)(k+1)}}{q^{(j+s-1)(k+1)}} \frac{q^{(i+s-1)k}}{q^{(j+s-1)k}} \right|, \end{aligned}$$

where  $s$  is any integer. Hence we obtain

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(bq; q)_k}{(q; q)_k} (aq^{r-2s+1})^k \left| \frac{\frac{q^{(i+s-1)k}}{(q; q)_{i+s-2}}}{\frac{q^{(j+s-1)k}}{(q; q)_{j+s-2}}} \frac{\frac{q^{(i+s-1)(k+1)}}{(q; q)_{i+s-2}}}{\frac{q^{(j+s-1)(k+1)}}{(q; q)_{j+s-2}}} \right| \\ &= -q^s \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \cdot \frac{q^{i-1} - q^{j-1}}{(q; q)_{i+s-2} (q; q)_{j+s-2}} \cdot \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}}. \end{aligned} \quad (5.14)$$

Set  $t_j^i$  and  $\alpha_j$  to be

$$t_j^i = \frac{q^{(i+s-1)j}}{(q; q)_{i+s-2}}, \quad \alpha_j = \frac{(bq; q)_j}{(q; q)_j} (aq^{r-2s+1})^j$$

for  $i \geq 1$  and  $j \geq 0$ . Let  $B = (\beta_j^i)_{i,j \geq 0}$  be the skew-symmetric matrix defined by

$$\beta_j^i = \begin{cases} \alpha_i & \text{if } j = i + 1 \text{ for } i = 0, 1, \dots, \\ -\alpha_j & \text{if } i = j + 1 \text{ for } j = 0, 1, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

If we take  $T = (t_j^i)_{1 \leq i \leq 2n, 0 \leq j}$  and  $B = (\beta_j^i)_{i,j \geq 0}$  in Theorem 2.3, then, by Proposition 2.4, we obtain

$$\begin{aligned} &\sum_{\mu} (aq^{r-2s+1})^{\sum_{k=1}^n \mu_{2k-1}} \prod_{k=1}^n \frac{(bq; q)_{\mu_{2k-1}}}{(q; q)_{\mu_{2k-1}}} \cdot \det T_{\mu}^{[2n]} \\ &= (-1)^n q^{ns} \left\{ \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \right\}^n \prod_{k=1}^{2n} \frac{1}{(q; q)_{k+s-2}} \cdot \text{Pf} (a_j^i)_{1 \leq i, j \leq 2n}, \end{aligned} \quad (5.15)$$

where the sum on the left-hand side runs over all  $2n$ -tuples  $\mu = (\mu_1, \mu_2, \dots, \mu_{2n})$  of integers such that  $0 \leq \mu_1 < \mu_2 < \dots < \mu_{2n}$  and  $\mu_2 = \mu_1 + 1, \dots, \mu_{2n} = \mu_{2n-1} + 1$ . Now we take the shape  $\lambda = (m, m-1, \dots, m-2n+1)$  for positive integers  $m, n$  such that  $m \geq 2n$  in Lemma 5.4. Then (5.11) implies that  $\text{GF}[\mathcal{T}_{\lambda, \mu}]$  equals

$$\det \left( \frac{q^{(m-i+1)\mu_j}}{(q; q)_{m-i}} \right)_{1 \leq i, j \leq 2n} = (-1)^n \det \left( \frac{q^{(i+m-2n)\mu_j}}{(q; q)_{i+m-2n-1}} \right)_{1 \leq i, j \leq 2n} \quad (5.16)$$

for a profile  $\mu = (\mu_1, \dots, \mu_{2n})$  where  $0 \leq \mu_1 < \dots < \mu_{2n}$ . Here the right-hand side is obtained from the left-hand side by reversing the order of row indices. Let  $s = m - 2n + 1$ . If we substitute (5.16) into (5.15) and use Corollary 3.2, then we obtain

$$\begin{aligned} & \sum_{\mu} (aq^{r-2s+1})^{\sum_{k=1}^n \mu_{2k-1}} \prod_{k=1}^n \frac{(bq; q)_{\mu_{2k-1}}}{(q; q)_{\mu_{2k-1}}} \cdot \text{GF}[\mathcal{T}_{\lambda, \mu}] \\ &= q^{ns} \left\{ \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \right\}^n \prod_{k=1}^{2n} \frac{1}{(q; q)_{k+s-2}} \\ & \times a^{n(n-1)} q^{n(n-1)(4n+1)/3+n(n-1)r} \prod_{k=1}^{n-1} (bq; q)_{2k} \prod_{k=1}^n \frac{(q; q)_{2k-1} (aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}}. \end{aligned}$$

This proves (5.12). If we put  $\mu = \nu + \epsilon_{2n}$  and use the fact that  $n(\lambda) = mn(2n-1) - \frac{1}{3}n(2n-1)(4n-1)$ , then we can prove (5.6) by a direct computation. The other identities can be proven similarly. The details are left to the reader.  $\square$

Theorem 5.2 treats only reverse plane partitions whose number of rows is even. We obtain the case where the number of rows equals  $2n-1$  from Corollary 4.4. Let  $\check{A} = (\check{a}_{ij}^i)_{0 \leq i, j}$  be the skew-symmetric matrix whose  $(i, j)$ -entry for  $0 \leq i < j$  equals

$$\check{a}_{ij}^i = \begin{cases} \frac{(aq; q)_{j+r-1}}{(abq^2; q)_{j+r-1}} & \text{if } i = 0 \text{ and } j \geq 1, \\ (q^{i-1} - q^{j-1}) \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} & \text{if } 1 \leq i < j. \end{cases}$$

Then it is easy to see that Corollary 4.4 implies that for  $n \geq 1$

$$\begin{aligned} \text{Pf}(\check{a}_{ij}^i)_{0 \leq i, j \leq 2n-1} &= a^{(n-1)^2} q^{n(n-1)(4n-5)/3+(n-1)^2r} \\ & \times \frac{(aq; q)_r}{(abq^2; q)_r} \prod_{k=1}^{n-1} \frac{(q; q)_{2k} (aq; q)_{2k+r} (bq; q)_{2k-1}}{(abq^2; q)_{4k+r-1} (abq^{2k+r}; q)_{2k-1}}. \end{aligned} \quad (5.17)$$



Let  $\check{P}_{n,r}(a, b; q)$  denote the right-hand side of (5.17). Then, more generally, from (4.14) and (4.15) we derive for  $n \geq 2$

$$\text{Pf}(\check{A}_{[0,2n-2],m-1}) = \frac{(q^{m-2n+1}; q)_{2n-2} (aq^{2n+r-1}; q)_{m-2n}}{(q; q)_{2n-2} (abq^{4n+r-3}; q)_{m-2n}} \check{P}_{n,r}(a, b; q), \quad (5.18)$$

$$\begin{aligned} \text{Pf}(\check{A}_{[0,2n-3],2n-1,m-1}) &= q \cdot \frac{(q^{m-2n}; q)_1 (q^{m-2n+2}; q)_{2n-3} (aq^{2n+r-1}; q)_{m-2n}}{(q; q)_{2n-2} (abq^{4n+r-5}; q)_1 (abq^{4n+r-3}; q)_{m-2n+1}} \\ &\quad \times f(2n-1, m-1, r) \check{P}_{n,r}(a, b; q). \end{aligned} \quad (5.19)$$

As an application of (5.17), (5.18) and (5.19), we can derive a similar identities in the case where the number of rows of the shapes is odd. Before we state our theorem we need a few definitions. Fix positive integers  $n$  and  $t$  such that  $1 \leq t \leq n$ . Let  $\mathcal{Q}_n^{(t)}$  denote the set of profiles  $\nu = (\nu_1, \dots, \nu_{2n-1})$  such that  $0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_{2n-1}$ ,  $\nu_{2k} = \nu_{2k-1}$  for  $k = 1, \dots, t-1$  and  $\nu_{2k+1} = \nu_{2k}$  for  $k = t, \dots, n-1$ . For  $\nu \in \mathcal{Q}_n^{(t)}$  and  $x, y \in \mathbb{Z}$  we let

$$\begin{aligned} \psi_{x,y}^{(t)}(\nu) &= (aq^x)^{(|\nu| - \nu_{2t-1})/2} (aq^y)^{\nu_{2t-1}} \\ &\quad \times \frac{(bq^{2t-1}; q)_{\nu_{2t-1}}}{(q^{2t-1}; q)_{\nu_{2t-1}}} \prod_{k=1}^{t-1} \frac{(bq^{2k}; q)_{\nu_{2k-1}-1}}{(q^{2k}; q)_{\nu_{2k-1}-1}} \prod_{k=t}^{n-1} \frac{(bq^{2k}; q)_{\nu_{2k}}}{(q^{2k}; q)_{\nu_{2k}}}, \end{aligned} \quad (5.20)$$

where  $|\nu| = \sum_{k=1}^{2n-1} \nu_k$ . Then we obtain the following theorem from (5.17), (5.18) and (5.19).

**Theorem 5.5.** Let  $r$  be an integer. For any positive integers  $m$  and  $n$  such that  $m \geq 2n-1$ , we fix the shape  $\lambda = (m, m-1, \dots, m-2n+2)$  of length  $2n-1$ . Then we have

$$\begin{aligned} &\sum_{t=1}^n (aq^{r+1})^{t-1} \frac{(bq; q)_{2(t-1)}}{(q; q)_{2(t-1)}} \sum_{\nu \in \mathcal{Q}_n^{(t)}} \psi_{r-2(m-2n)-3, r-(m-2n)-1}^{(t)}(\nu) \text{GF}[\mathcal{R}_{\lambda, \nu}] \\ &= \prod_{k=1}^{2n-1} \frac{(q; q)_{k-1}}{(q; q)_{k+m-2n}} \cdot R_{n,r}(a, b; q), \end{aligned} \quad (5.21)$$

where

$$R_{n,r}(a, b; q) = \left\{ \frac{(abq^2; q)_{\infty}}{(aq; q)_{\infty}} \right\}^n \cdot \frac{(aq; q)_r}{(abq^2; q)_r} \cdot \prod_{k=1}^{n-1} \frac{(aq; q)_{2k+r}}{(abq^2; q)_{4k+r-1} (abq^{2k+r}; q)_{2k-1}}.$$

More generally, if  $\lambda = (l, m-1, m-2, m-3, \dots, m-2n+2)$  where  $l \geq m$  and  $n \geq 2$ , then we have

$$\begin{aligned} & \sum_{t=1}^n (aq^{r+1})^{t-1} \frac{(bq; q)_{2(t-1)}}{(q; q)_{2(t-1)}} \sum_{\nu \in \mathcal{Q}_n^{(t)}} \psi_{r-2(m-2n)-3, r-(m-2n)-1}^{(t)}(\nu) \text{GF}[\mathcal{R}_{\lambda, \nu}] \\ &= \frac{\prod_{k=1}^{2n-1} (q; q)_{k-1}}{(q; q)_{l-1} \prod_{k=1}^{2n-2} (q; q)_{k+m-2n}} \cdot \frac{(q^{l-m+1}; q)_{2n-2} (aq^{2n+r-1}; q)_{l-m}}{(q; q)_{2n-2} (abq^{4n+r-3}; q)_{l-m}} R_{n,r}(a, b; q), \end{aligned} \quad (5.22)$$

and if  $\lambda = (l, m, m-2, m-3, \dots, m-2n+2)$  where  $l > m$  and  $n \geq 2$ , then we have

$$\begin{aligned} & \sum_{t=1}^n (aq^{r+1})^{t-1} \frac{(bq; q)_{2(t-1)}}{(q; q)_{2(t-1)}} \sum_{\nu \in \mathcal{Q}_n^{(t)}} \psi_{r-2(m-2n)-3, r-(m-2n)-1}^{(t)}(\nu) \text{GF}[\mathcal{R}_{\lambda, \nu}] \\ &= \frac{f(2n-1, l-m+2n-1, r) \prod_{k=1}^{2n-1} (q; q)_{k-1}}{(q; q)_{l-1} (q; q)_{m-1} \prod_{k=1}^{2n-3} (q; q)_{k+m-2n}} \\ & \quad \times \frac{(q^{l-m}; q)_1 (q^{l-m+2}; q)_{2n-3} (aq^{2n+r-1}; q)_{l-m}}{(q; q)_{2n-2} (abq^{4n+r-5}; q)_1 (abq^{4n+r-3}; q)_{l-m+1}} R_{n,r}(a, b; q). \end{aligned} \quad (5.23)$$

To prove this theorem, define a matrix  $T = (t_j^i)_{i \geq 0, j \geq -1}$  by

$$t_j^i = \begin{cases} 1 & \text{if } i = 0 \text{ and } j = -1, \\ \frac{q^{(i+s-1)j}}{(q; q)_{i+s-2}} & \text{if } i \geq 1 \text{ and } j \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and a skew-symmetric matrix  $B = (\beta_j^i)_{-1 \leq i < j}$  by

$$\beta_j^i = \begin{cases} \frac{(bq; q)_j}{(q; q)_j} (aq^{r-s+1})^j & \text{if } i = -1 \text{ and } j \geq 0, \\ \frac{(bq; q)_i}{(q; q)_i} (aq^{r-2s+1})^i & \text{if } 0 \leq i < j \text{ and } i+1 = j, \\ 0 & \text{otherwise,} \end{cases}$$

where  $s = m - 2n + 2$ . A similar reasoning as in the proof of Theorem 5.2 works to prove these identities. We omit the details.

## 6 Open problems

In this section we formulate several conjectures for the Pfaffians of certain sequences related to Catalan numbers based on the computer experiments. The *Al-Salam-Carlitz polynomials* [3] are defined by

$$U_n^{(a)}(x; q) = (-1)^n q^{\binom{n}{2}} {}_2\phi_1 \left( \begin{matrix} q^{-n}, ax^{-1} \\ 0 \end{matrix}; q, qx \right).$$

Let  $L$  be the linear functional with respect to which  $U_n^{(a)}(x; q)$  are orthogonal. Then the  $n$ th moment has the expression [9, 16, 19]:

$$G_n(a; q) = L(x^n) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q a^k,$$

where  $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$ .

**Conjecture 6.1.** Let  $n \geq 1$  be an integer. Then the following identities would hold:

$$\begin{aligned} & \text{Pf} \left( (q^{i-1} - q^{j-1}) G_{i+j-3}(a; q) \right)_{1 \leq i, j \leq 2n} \\ &= a^{n(n-1)} q^{\frac{1}{3} \lfloor n/2 \rfloor (16 \lfloor n/2 \rfloor^2 - 1) - (-1)^n 4 \lfloor n/2 \rfloor^2 - 2 \lfloor n/2 \rfloor \cdot \lfloor (n-1)/2 \rfloor} \prod_{k=1}^n (q; q)_{2k-1}, \end{aligned} \quad (6.1)$$

$$\begin{aligned} & \text{Pf} \left( (q^{i-1} - q^{j-1}) G_{i+j-2}(a; q) \right)_{1 \leq i, j \leq 2n} \\ &= a^{n(n-1)} q^{\frac{1}{3} \lfloor n/2 \rfloor (16 \lfloor n/2 \rfloor^2 - 1) - (-1)^n 4 \lfloor n/2 \rfloor^2} \prod_{k=1}^n (q; q)_{2k-1} \sum_{k=0}^n q^{\lfloor (n-2k)^2/2 \rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} a^k. \end{aligned} \quad (6.2)$$

Here  $\lfloor x \rfloor$  denotes the largest integer which is not greater than  $x$ , and we use the convention that  $G_{-1}(a; q) = 0$  which can in fact be assigned to any value.

There are several well-known numbers related to lattice path enumeration ([2, 25]). Let  $M_n = \sum_{k=0}^n \binom{n}{2k} C_k$  denote the *Motzkin numbers*,  $D_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k}$  the *central Delannoy numbers*, and  $S_n = \sum_{k=0}^n \binom{n+k}{2k} C_k$  *Schröder*

numbers. Finally, the number  $N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$  is known as a *Narayana number*, and

$$N_n(a) = \sum_{k=0}^n \frac{1}{n} \binom{n}{k} \binom{n}{k-1} a^k$$

is known as the  $n$ th *Narayana polynomial*, which is the moment sequence of a *generalized Chebyshev polynomials* of the first kind. Here we use the convention that  $N_0(a) = 1$ .

**Conjecture 6.2.** Let  $n \geq 1$  be an integer. Then the following identities would hold:

$$\text{Pf} \left( (j-i)M_{i+j-3} \right)_{1 \leq i, j \leq 2n} = \prod_{k=0}^{n-1} (4k+1), \quad (6.3)$$

$$\text{Pf} \left( (j-i)D_{i+j-3} \right)_{1 \leq i, j \leq 2n} = 2^{n^2-1} (2n-1) \prod_{k=1}^{n-1} (4k-1), \quad (6.4)$$

$$\text{Pf} \left( (j-i)S_{i+j-2} \right)_{1 \leq i, j \leq 2n} = 2^{n^2} \prod_{k=0}^{n-1} (4k+1), \quad (6.5)$$

$$\text{Pf} \left( (j-i)N_{i+j-2}(a) \right)_{1 \leq i, j \leq 2n} = a^{n^2} \prod_{k=0}^{n-1} (4k+1). \quad (6.6)$$

Note that  $C_n = N_n(1)$ ,  $S_n = N_n(2)$  and

$$M_{n-1} = \left( \frac{1 - \sqrt{-3}}{2} \right)^{n+1} N_n \left( \frac{-1 + \sqrt{-3}}{2} \right).$$

Hence, if one could prove (6.6), then one would have proven (6.3) and (6.5) as corollaries.

Let  $a_n = \frac{1}{2n+1} \binom{3n}{n} = \frac{1}{3n+1} \binom{3n+1}{n}$ . In [11] Gessel and Xin prove that  $\det(a_{i+j-1})_{1 \leq i, j \leq n}$  equals the number of  $(2n+1) \times (2n+1)$  alternating sign matrices that are invariant under vertical reflection. We propose the following conjecture concerning this sequence.

**Conjecture 6.3.** Let  $a_n$  be as above. Then the following identity would hold:

$$\text{Pf}((j-i)a_{i+j-1})_{1 \leq i, j \leq 2n} = \frac{1}{2^n} \prod_{k=1}^n \frac{(12k-6)!(4k-3)!(3k-1)!}{(8k-6)!(8k-3)!(3k-2)!}. \quad (6.7)$$

## Appendix: Creative telescoping

In this appendix we state an alternative proof of (4.2) and (4.4) by Zeilberger's creative telescoping [20, 24]. In this case the certificates are extremely simple, and we can check the computation by hand. We note that one can prove (4.5) similarly, but the certificate for (4.5) is a little more complicated.

By replacing  $q^i$  by  $c$ , the equations (4.2) and (4.4) are generalized as

$$\begin{aligned} \sum a^{k-1} q^{k(k-1)+1} \cdot \frac{(abq^{2k}; q)_1 (abq^2; q)_{k-2} (bq, cq^{-k+1}, q^{j-k+1}; q)_{k-1}}{(q, aq, abcq, abq^{j+1}; q)_k} \\ \times h_k(j; a, b, c, q) = \frac{(ac, abq^2; q)_{j-1}}{(aq, abcq; q)_{j-1}}, \end{aligned} \quad (6.8)$$

where the sum on the left-hand side runs over odd positive integers or even nonnegative integers, and  $h_k(j; a, b, c, q)$  is set to be

$$\begin{aligned} h_k(j; a, b, c, q) &= (1 - q^k)(1 - aq^k) \\ &\times \left\{ q^{-k}(1 + abq^{2k})(1 + abcq^{j-1}) - ab(1 + q)(cq^{-1} + q^{j-1}) \right\} \\ &+ aq^{k-1}(1 - b)(1 - cq^{-k})(1 - q^{j-k})(1 - abq^{2k+1}). \end{aligned} \quad (6.9)$$

Let

$$\begin{aligned} F(j, k) &= a^{k-1} c^{k-1} q^{j(k-1)+1} \cdot \frac{(abq^{2k}; q)_1 (abq^2; q)_{k-2} (bq, q/c, q^{1-j}; q)_{k-1}}{(q, aq, abcq, abq^{j+1}; q)_k} \\ &\times h_k(j; a, b, c, q). \end{aligned} \quad (6.10)$$

Hereafter we use the notation that  $F^{(o)}(j, k) = F(j, 2k-1)$  and  $F^{(e)}(j, k) = F(j, 2k-2)$ . Further we set  $T^{(o)}(j, k) = T(j, 2k-1)$  and  $T^{(e)}(j, k) = T(j, 2k-2)$ , where

$$T(j, k) = F(j, k) - \frac{(1 - aq^j)(1 - abcq^j)}{(1 - abq^{j+1})(1 - acq^{j-1})} F(j+1, k). \quad (6.11)$$

Let us define  $P^{(x)}(j, k)$ ,  $Q^{(x)}(j, k)$  and  $R^{(x)}(j, k)$  for  $x = o, e$  by  $P^{(o)}(j, k) = P(j, 2k-1)$ ,  $Q^{(o)}(j, k) = Q(j, 2k-1)$ ,  $R^{(o)}(j, k) = R(j, 2k-1)$ ,  $P^{(e)}(j, k) =$

$P(j, 2k - 2)$ ,  $Q^{(e)}(j, k) = Q(j, 2k - 2)$ , and  $R^{(e)}(j, k) = R(j, 2k - 2)$ , where

$$\begin{aligned}
P(j, k) &= a^2 c^2 q^{2j} (1 - abq^k)(1 - abq^{k+1})(1 - bq^k)(1 - bq^{k+1}) \\
&\quad \times (1 - q^k/c)(1 - q^{k+1}/c)(1 - q^{k-j-1})(1 - q^{k-j}), \\
Q(j, k) &= (1 - q^{k+1})(1 - q^{k+2})(1 - aq^{k+1})(1 - aq^{k+2})(1 - abcq^{k+1}) \\
&\quad \times (1 - abcq^{k+2})(1 - abq^{j+k+2})(1 - abq^{j+k+3}), \\
R(j, k) &= (1 - abq^{2k}) \left\{ (abq^{j+k+1}, q^{k-j-1}; q)_1 h_k(j; a, b, c, q) \right. \\
&\quad \left. - \frac{q^{k-1} (1 - aq^j) (1 - abcq^j)}{(1 - abq^{j+1}) (1 - acq^{j-1})} (abq^{j+1}, q^{-j}; q)_1 h_k(j+1; a, b, c, q) \right\}.
\end{aligned}$$

By direct computation, we see that

$$\frac{T^{(x)}(j, k+1)}{T^{(x)}(j, k)} = \frac{P^{(x)}(j, k)}{Q^{(x)}(j, k)} \cdot \frac{R^{(x)}(j, k+1)}{R^{(x)}(j, k)} \quad (6.12)$$

holds for  $x = o, e$ . We define  $\Lambda^{(x)}(j, k)$  by

$$\Lambda^{(x)}(j, k) = \frac{Q^{(x)}(j, k-1)T^{(x)}(j, k)}{R^{(x)}(j, k)} X^{(x)}(j, k) \quad (6.13)$$

for  $x = o, e$ , where  $X^{(o)}(j, k) = X(j, 2k - 1)$  and  $X^{(e)}(j, k) = X(j, 2k - 2)$ , with

$$X(j, k) = -\frac{q^{-k}}{1 - acq^{j-1}}. \quad (6.14)$$

**Lemma 6.4.** Let  $\Lambda^{(x)}(j, k)$  be as above for  $x = o, e$ . Then we have

$$T^{(x)}(j, k) = \Lambda^{(x)}(j, k+1) - \Lambda^{(x)}(j, k) \quad (6.15)$$

for  $x = o, e$ .

**Proof.** If one use (6.12) and (6.13) then (6.15) reduces to the following identity:

$$P^{(x)}(j, k)X^{(x)}(j, k+1) - Q^{(x)}(j, k-1)X^{(x)}(j, k) = R^{(x)}(j, k), \quad (6.16)$$

This can be checked by direct computation.  $\square$

Because of  $\Lambda^{(x)}(j, 1) = 0$  for  $x = o, e$ , by summing (6.15) over all positive integers, we obtain

$$\sum_{k \geq 1} F^{(x)}(j+1, k) = \frac{(1 - abq^{j+1})(1 - acq^{j-1})}{(1 - aq^j)(1 - abcq^j)} \sum_{k \geq 1} F^{(x)}(j, k) \quad (6.17)$$

for  $x = o, e$ . Since  $F(1, 0) = F(1, 1) = 1$  and  $F(1, k) = 0$  for  $k \geq 2$ , we have

$$\sum_{k \geq 1} F^{(x)}(1, k) = 1 \quad (6.18)$$

for  $x = o, e$ . Hence we obtain the desired identity (6.8) from (6.17). This gives the second proof of Theorem 3.1 and Theorem 4.3.

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